

Directed Information on Abstract Spaces: Properties and Extremum Problems

Charalambos D. Charalambous and Photios A. Stavrou

ECE Department, University of Cyprus, Green Park, Aglantzias 91,

P.O. Box 20537, 1687, Nicosia, Cyprus

e-mail: *chadcha@ucy.ac.cy, stavrou.fotios@ucy.ac.cy*

Abstract—This paper describes a framework in which directed information is defined on abstract spaces. The framework is employed to derive properties of directed information such as convexity, concavity, lower semicontinuity, by using the topology of weak convergence of probability measures on Polish spaces. Two extremum problems of directed information related to capacity of channels with memory and feedback, and causal and sequential rate distortion are extensively analyzed showing existence of maximizing and minimizing solutions, respectively.

I. INTRODUCTION

Directed information from a sequence of Random Variables (RV's) $X^n \triangleq \{X_0, X_1, \dots, X_n\} \in \mathcal{X}_{0,n} \triangleq \times_{i=0}^n \mathcal{X}_i$, to another sequence $Y^n \triangleq \{Y_0, Y_1, \dots, Y_n\} \in \mathcal{Y}_{0,n} \triangleq \times_{i=0}^n \mathcal{Y}_i$ is often defined via [1], [2]

$$I(X^n \rightarrow Y^n) \triangleq \sum_{i=0}^n I(X^i; Y_i | Y^{i-1}) \quad (1)$$

$$= \sum_{i=0}^n \int_{\mathcal{X}_{0,i} \times \mathcal{Y}_{0,i}} \log \left(\frac{P_{Y_i | Y^{i-1}, X^i}(dy_i | y^{i-1}, x^i)}{P_{Y_i | Y^{i-1}}(dy_i | y^{i-1})} \right) \times P_{X^i, Y^i}(dx^i, dy^i) \quad (2)$$

$$\equiv \mathbb{I}_{X^n \rightarrow Y^n}(P_{X_i | X^{i-1}, Y^{i-1}}, P_{Y_i | Y^{i-1}, X^i} : i = 0, \dots, n) \quad (3)$$

Since the joint distribution in (2) is decomposed via $P_{X^i, Y^i}(dx^i, dy^i) = \otimes_{j=0}^i P_{X_j | X^{j-1}, Y^{j-1}}(dx_j | x^{j-1}, y^{j-1}) \otimes P_{Y_j | Y^{j-1}, X^j}(dy_j | y^{j-1}, x^j)$, the notation $\mathbb{I}_{X^n \rightarrow Y^n}(\cdot, \cdot)$ denotes the functional dependence on two collections of causal conditional distributions $\{P_{X_i | X^{i-1}, Y^{i-1}}(\cdot | \cdot, \cdot) : i = 0, 1, \dots, n\}$ and $\{P_{Y_i | Y^{i-1}, X^i}(\cdot | \cdot, \cdot) : i = 0, 1, \dots, n\}$.

In information theory, directed information (1)-(3) or its variants are used to characterize capacity of channels with memory and feedback [3], [4], [5], lossy data compression with feedforward information at the decoder [6], lossy data compression of sequential codes [4], lossy data compression of causal codes [7], and capacity of networks such as the two-way channel, multiple access channel [8], [9], etc. The previous references derive coding theorems based on *a*) stationary ergodic processes $\{(X_i, Y_i)\}_{i=0}^\infty$, *b*) Dobrushin's stability of the information density $\log \otimes_{i=0}^n \frac{P_{Y_i | Y^{i-1}, X^i}(dy_i | y^{i-1}, x^i)}{P_{Y_i | Y^{i-1}}(dy_i | y^{i-1})}$, and *c*) via information spectrum methods [10].

Capacity with Feedback. Based on *a*) or *b*) the operational

definition of channels with memory and feedback is given by

$$C^f(P) \triangleq \lim_{n \rightarrow \infty} \sup_{\bar{P}_{X^n | Y^{n-1}}(\cdot | \cdot) \in \bar{\mathcal{P}}(P)} \frac{1}{n+1} (X^n \rightarrow Y^n)$$

where $\bar{\mathcal{P}}(P)$ denotes the power constraint set, and

$$\bar{P}_{X^n | Y^{n-1}}(dx^n | dy^{n-1}) \triangleq \otimes_{i=0}^n P_{X_i | X^{i-1}, Y^{i-1}}(dx_i | x^{i-1}, y^{i-1})$$

Sequential and Causal Rate Distortion. Based *a*) or *b*) the operational definition of sequential and causal rate distortion function is given by expression

$$R^c(D) \triangleq \lim_{n \rightarrow \infty} \inf_{\bar{P}_{Y^n | X^n}(\cdot | \cdot) \in \bar{\mathcal{Q}}(D)} \frac{1}{n+1} (X^n \rightarrow Y^n)$$

where $\bar{\mathcal{Q}}(D)$ is the distortion fidelity constraint and

$$\bar{P}_{Y^n | X^n}(dy^n | x^n) \triangleq \otimes_{i=0}^n P_{Y_i | Y^{i-1}, X^i}(dy_i | y^{i-1}, x^i), \\ P_{X_i | X^{i-1}, Y^{i-1}}(dx_i | x^{i-1}, y^{i-1}) = P_{X_i | X^{i-1}}(dx_i | x^{i-1}) - a.s.$$

The complete characterization and properties of the above extremum problems requires extensive analysis of the functional $\mathbb{I}_{X^n \rightarrow Y^n}(\cdot, \cdot)$ as defined in (3). This is analogous to capacity of channels without feedback which involves maximization of mutual information $I(X^n; Y^n)$ over the power constraint set, and to classical rate distortion function which involves minimization of mutual information $I(X^n; Y^n)$ over the fidelity constraint. However, mutual information $I(X^n; Y^n) \equiv \mathbb{I}_{X^n, Y^n}(P_{X^n}, P_{Y^n | X^n})$, inherits from its information divergence definition $I(X^n; Y^n) = \mathbb{D}(P_{X^n, Y^n} || P_{X^n} \times P_{Y^n})$, several important functional properties such as convexity, concavity, lower semicontinuity, etc. These properties are vital both for finite alphabet spaces [11], as well as abstract alphabet spaces [12], [13]. The difficulty associated with directed information $I(X^n \rightarrow Y^n)$, rises from the fact that this information measure (1)-(3) is a functional $\mathbb{I}_{X^n \rightarrow Y^n}(\cdot, \cdot)$ of the collection of conditional distributions $\{P_{X_i | X^{i-1}, Y^{i-1}}(\cdot | \cdot, \cdot) : i = 0, 1, \dots, n\}$ and $\{P_{Y_i | Y^{i-1}, X^i}(\cdot | \cdot, \cdot) : i = 0, 1, \dots, n\}$. The objective of this paper is to address the following questions, when $\mathcal{X}_{0,n}$ and $\mathcal{Y}_{0,n}$ are complete separable metric spaces (Polish spaces).

1. Is there an equivalent directed information definition expressed via information divergence $\mathbb{D}(\cdot || \cdot)$ as a functional of two appropriate conditional distributions $\mathbf{P}(\cdot | \mathbf{y})$ on $\mathcal{X}^{\mathbb{N}} \triangleq \times_{i=0}^\infty \mathcal{X}_i$ for $\mathbf{y} = (y_0, y_1, \dots) \in \mathcal{Y}^{\mathbb{N}} \triangleq \times_{i=0}^\infty \mathcal{Y}_i$

and $\mathbf{Q}(\cdot|\mathbf{x})$ on $\mathcal{Y}^{\mathbb{N}}$ for $\mathbf{x} \in \mathcal{X}^{\mathbb{N}}$ which uniquely define $\{P_{X_i|X^{i-1}, Y^{i-1}} : i = 0, 1, \dots\}$ and $\{P_{Y_i|Y^{i-1}, X^i} : i = 0, 1, \dots\}$, respectively, and vice-versa?

2. Is directed information convex and concave functional with respect to the conditional distributions $\mathbf{P}(\cdot|\mathbf{y})$ and $\mathbf{Q}(\cdot|\mathbf{x})$?
3. Is directed information a lower semicontinuous functional of the conditional distributions $\mathbf{P}(\cdot|\mathbf{y})$ and $\mathbf{Q}(\cdot|\mathbf{x})$?
4. What are appropriate conditions for the abstract spaces $\mathcal{X}_{0,n}$ and $\mathcal{Y}_{0,n}$ on which existence of the maximizing encoder admissible distributions and minimizing distortion admissible distribution can be sought?

Complete answers to the above questions are provided by invoking the topology of weak convergence of probability measures on Polish spaces and Prohorov's theorems. The derivation are outlined since they are quite lengthy.

II. CAUSAL CHANNELS ON ABSTRACT SPACES

In this section, the aim is to establish two equivalent definitions of conditional distributions or basic processes, which define any probabilistic channel with causal feedback, that relate causally the input-output behavior of any channel. This formulation is necessary to investigate questions 1–4.

Let $\mathbb{N} \triangleq \{0, 1, 2, \dots\}$, and $\mathbb{N}^n \triangleq \{0, 1, 2, \dots, n\}$. Introduce two sequence of spaces $\{(\mathcal{X}_n, \mathcal{B}(\mathcal{X}_n)) : n \in \mathbb{N}\}$ and $\{(\mathcal{Y}_n, \mathcal{B}(\mathcal{Y}_n)) : n \in \mathbb{N}\}$, where $\mathcal{X}_n, \mathcal{Y}_n, n \in \mathbb{N}$ are topological spaces, and $\mathcal{B}(\mathcal{X}_n)$ and $\mathcal{B}(\mathcal{Y}_n)$ are Borel σ -algebras of subsets of \mathcal{X}_n and \mathcal{Y}_n , respectively. Points in $\mathcal{X}^{\mathbb{N}} \triangleq \times_{n \in \mathbb{N}} \mathcal{X}_n$, $\mathcal{Y}^{\mathbb{N}} \triangleq \times_{n \in \mathbb{N}} \mathcal{Y}_n$ are denoted by $\mathbf{x} \triangleq \{x_0, x_1, \dots\} \in \mathcal{X}^{\mathbb{N}}$, $\mathbf{y} \triangleq \{y_0, y_1, \dots\} \in \mathcal{Y}^{\mathbb{N}}$, respectively, while their restrictions to finite coordinates by $x^n \triangleq \{x_0, x_1, \dots, x_n\} \in \mathcal{X}_{0,n}$, $y^n \triangleq \{y_0, y_1, \dots, y_n\} \in \mathcal{Y}_{0,n}$, for $n \in \mathbb{N}$.

Let $\mathcal{B}(\mathcal{X}^{\mathbb{N}}) \triangleq \odot_{i \in \mathbb{N}} \mathcal{B}(\mathcal{X}_i)$ denote the σ -algebra in $\mathcal{X}^{\mathbb{N}}$ generated by cylinder sets $\{\mathbf{x} = (x_0, x_1, \dots) \in \mathcal{X}^{\mathbb{N}} : x_0 \in A_0, x_1 \in A_1, \dots, x_n \in A_n\}$, $A_i \in \mathcal{B}(\mathcal{X}_i)$, $0 \leq i \leq n, n \geq 1$, and similarly for $\mathcal{B}(\mathcal{Y}^{\mathbb{N}}) \triangleq \odot_{i \in \mathbb{N}} \mathcal{B}(\mathcal{Y}_i)$.

Hence, $\mathcal{B}(\mathcal{X}_{0,n})$ and $\mathcal{B}(\mathcal{Y}_{0,n})$ denote the σ -algebras of cylinder sets in $\mathcal{X}^{\mathbb{N}}$ and $\mathcal{Y}^{\mathbb{N}}$, respectively, with bases over $A_i \in \mathcal{B}(\mathcal{X}_i)$, and $B_i \in \mathcal{B}(\mathcal{Y}_i)$, $0 \leq i \leq n$, respectively.

Backward or Feedback Channel. Suppose for each $n \in \mathbb{N}$, the distributions $\{p_n(dx_n|x^{n-1}, y^{n-1}) : n \in \mathbb{N}\}$ with $p_0(dx_0|x^{-1}, y^{-1}) \triangleq p_0(x_0)$ satisfy the following conditions.

- i) For $n \in \mathbb{N}$, $p_n(\cdot|x^{n-1}, y^{n-1})$ is a probability measure on $\mathcal{B}(\mathcal{X}_n)$;
- ii) For $n \in \mathbb{N}$, $A_n \in \mathcal{B}(\mathcal{X}_n)$, $p_n(A_n|x^{n-1}, y^{n-1})$ is $\odot_{i=0}^{n-1} \mathcal{B}(\mathcal{X}_i) \odot \mathcal{B}(\mathcal{Y}_i)$ -measurable in $x^{n-1} \in \mathcal{X}_{0,n-1}$, $y^{n-1} \in \mathcal{Y}_{0,n-1}$.

Given the collection $\{p_n(dx_n|x^{n-1}, y^{n-1}) : n \in \mathbb{N}\}$ satisfying conditions i), ii), one can construct a family of distributions on $(\mathcal{X}^{\mathbb{N}}, \mathcal{B}(\mathcal{X}^{\mathbb{N}})) \triangleq (\times_{i \in \mathbb{N}} \mathcal{X}_i, \odot_{i \in \mathbb{N}} \mathcal{B}(\mathcal{X}_i))$ as follows.

Let $C \in \mathcal{B}(\mathcal{X}_{0,n})$ be a cylinder set of the form $C \triangleq \{\mathbf{x} \in \mathcal{X}^{\mathbb{N}} : x_0 \in C_0, x_1 \in C_1, \dots, x_n \in C_n\}$, $C_i \in \mathcal{B}(\mathcal{X}_i)$, $0 \leq i \leq n$.

Define a family of measures $\mathbf{P}(\cdot|\mathbf{y})$ on $\mathcal{B}(\mathcal{X}^{\mathbb{N}})$ by

$$\mathbf{P}(C|\mathbf{y}) \triangleq \int_{C_0} p_0(dx_0) \dots \int_{C_n} p_n(dx_n|x^{n-1}, y^{n-1}) \quad (4)$$

$$\equiv \overleftarrow{P}_{0,n}(C_{0,n}|y^{n-1}), \quad C_{0,n} = \times_{i=0}^n C_i \quad (5)$$

The notation $\overleftarrow{P}_{0,n}(\cdot|y^{n-1})$ is used to denote the restriction of the measure $\mathbf{P}(\cdot|\mathbf{y})$ on cylinder sets $C \in \mathcal{B}(\mathcal{X}_{0,n})$, for $n \in \mathbb{N}$. Thus, if conditions i) and ii) hold then for each $\mathbf{y} \in \mathcal{Y}^{\mathbb{N}}$, the right hand side of (4) defines a consistent family of finite-dimensional distribution on $(\mathcal{X}^{\mathbb{N}}, \mathcal{B}(\mathcal{X}^{\mathbb{N}}))$, and hence there exists a unique measure on $(\mathcal{X}^{\mathbb{N}}, \mathcal{B}(\mathcal{X}^{\mathbb{N}}))$, from which $p_n(dx_n|x^{n-1}, y^{n-1})$ is obtained. This leads to the first, usual definition of a feedback channel, as a family of functions $p_n(dx_n|x^{n-1}, y^{n-1})$ satisfying conditions i) and ii).

An alternative, equivalent definition of a feedback channel is established as follows. Introduce the assumption

iii) $\{\mathcal{X}_n : n \in \mathbb{N}\}$ are complete separable metric spaces (Polish Spaces) and $\{\mathcal{B}(\mathcal{X}_n) : n \in \mathbb{N}\}$ are the σ -algebras of Borel sets.

Consider a family of measures $\mathbf{P}(\cdot|\mathbf{y})$ on $(\mathcal{X}^{\mathbb{N}}, \mathcal{B}(\mathcal{X}^{\mathbb{N}}))$ satisfying the following consistency condition.

C1: If $E \in \mathcal{B}(\mathcal{X}_{0,n})$, then $\mathbf{P}(E|\mathbf{y})$ is $\mathcal{B}(\mathcal{Y}_{0,n-1})$ -measurable function of $\mathbf{y} \in \mathcal{Y}^{\mathbb{N}}$.

Then, by assumption iii), for any family of measures $\mathbf{P}(\cdot|\mathbf{y})$ satisfying **C1** one can construct a collection of versions of conditional distributions $\{p_n(dx_n|x^{n-1}, y^{n-1}) : n \in \mathbb{N}\}$ satisfying conditions i) and ii) which are connected with $\mathbf{P}(\cdot|\mathbf{y})$ via relation (4).

Therefore, for Polish Spaces $\{\mathcal{X}_n : n \in \mathbb{N}\}$ the second definition is given by a family of measures $\mathbf{P}(\cdot|\mathbf{y})$ on $(\mathcal{X}^{\mathbb{N}}, \mathcal{B}(\mathcal{X}^{\mathbb{N}}))$ depending parametrically on $\mathbf{y} \in \mathcal{Y}^{\mathbb{N}}$ and satisfying the consistency condition **C1**.

The point to be made here is that the second equivalent definition of a feedback channel, together with similar definition for the forward channel is convenient to define directed information via relative entropy, similar to the mutual information definition, and extend well-known functional properties of mutual information to directed information.

Forward Channel. The previous methodology is repeated for the collection of functions $\{q_n(dy_n|y^{n-1}, x^n) : n \in \mathbb{N}\}$ which satisfy the following conditions.

iv) For $n \in \mathbb{N}$, $q_n(\cdot|y^{n-1}, x^n)$ is a probability measure on $\mathcal{B}(\mathcal{Y}_n)$;

v) For $n \in \mathbb{N}$, $B_n \in \mathcal{B}(\mathcal{Y}_n)$, $q_n(B_n|y^{n-1}, x^n)$ is $\odot_{i=0}^{n-1} \mathcal{B}(\mathcal{Y}_i) \odot \mathcal{B}(\mathcal{X}_i)$ -measurable function of $x^n \in \mathcal{X}_{0,n}$, $y^{n-1} \in \mathcal{Y}_{0,n-1}$.

Similarly as before, given a cylinder set $D \in \mathcal{B}(\mathcal{Y}_{0,n})$ of the form $D \triangleq \{\mathbf{y} \in \mathcal{Y}^{\mathbb{N}} : y_0 \in D_0, y_1 \in D_1, \dots, y_n \in D_n\}$, $D_i \in \mathcal{B}(\mathcal{Y}_i)$, $0 \leq i \leq n$, define a family of measures on $\mathcal{B}(\mathcal{Y}^{\mathbb{N}})$ by

$$\mathbf{Q}(D|\mathbf{x}) \triangleq \int_{D_0} q_0(dy_0|x_0) \dots \int_{D_n} q_n(dy_n|y^{n-1}, x^n) \quad (6)$$

$$\equiv \overrightarrow{Q}_{0,n}(D_{0,n}|x^n), \quad D_{0,n} = \times_{i=0}^n D_i \quad (7)$$

For each $\mathbf{x} \in \mathcal{X}^{\mathbb{N}}$ the right hand side of (6) defines a consistent family of finite dimensional distribution, hence there

exists a unique measure on $(\mathcal{Y}^{\mathbb{N}}, \mathcal{B}(\mathcal{Y}^{\mathbb{N}}))$ for which the family of distributions $\{q_n(dy_n|y^{n-1}, x^n) : n \in \mathbb{N}\}$ is obtained. Introduced the assumption

vi) $\{\mathcal{Y}_n : n \in \mathbb{N}\}$ are Polish Spaces and $\{\mathcal{B}(\mathcal{Y}_n) : n \in \mathbb{N}\}$ are the σ -algebras of Borel sets.

Consider a family of measures $\mathbf{Q}(D|\mathbf{x})$ satisfying the following consistency condition.

C2: If $F \in \mathcal{B}(\mathcal{Y}_{0,n})$, then $\mathbf{Q}(F|\mathbf{x})$ is $\mathcal{B}(\mathcal{X}_{0,n})$ -measurable function of $x \in \mathcal{X}^{\mathbb{N}}$.

Then, by assumption **vi)**, for any family of measures $\mathbf{Q}(\cdot|\mathbf{x})$ on $(\mathcal{Y}^{\mathbb{N}}, \mathcal{B}(\mathcal{Y}^{\mathbb{N}}))$ satisfying consistency condition **C2** one can construct a collection of functions $\{q_n(dy_n|y^{n-1}, x^n) : n \in \mathbb{N}\}$ satisfying conditions **iv)** and **v)** which are connected with $\mathbf{Q}(\cdot|\mathbf{x})$ via relation (6).

Given the basic measures $\mathbf{P}(\cdot|\mathbf{y})$ on $\mathcal{X}^{\mathbb{N}}$ and $\mathbf{Q}(\cdot|\mathbf{x})$ on $\mathcal{Y}^{\mathbb{N}}$ satisfying consistency condition **C1** and **C2**, respectively, construct the collections of conditional distributions as follows.

Let $A^{(n)} = \{\mathbf{x} : x_n \in A\}$, $A \in \mathcal{B}(\mathcal{X}_n)$ and $B^{(n)} = \{\mathbf{y} : y_n \in B\}$, $B \in \mathcal{B}(\mathcal{Y}_n)$. In addition, let $\mathbf{P}(A^{(n)}|\mathbf{y}|\mathcal{B}(\mathcal{X}_{0,n-1}))$ denote the conditional probability of $A^{(n)}$ with respect to $\mathcal{B}(\mathcal{X}_{0,n-1})$ calculated on the probability space $(\mathcal{X}^{\mathbb{N}}, \mathcal{B}(\mathcal{X}^{\mathbb{N}}), \mathbf{P}(\cdot|\mathbf{y}))$, and $\mathbf{Q}(B^{(n)}|\mathbf{x}|\mathcal{B}(\mathcal{Y}_{0,n-1}))$ denote the conditional probability of $B^{(n)}$ with respect to $\mathcal{B}(\mathcal{Y}_{0,n-1})$ calculated on the probability space $(\mathcal{Y}^{\mathbb{N}}, \mathcal{B}(\mathcal{Y}^{\mathbb{N}}), \mathbf{Q}(\cdot|\mathbf{x}))$. Then

$$\begin{aligned} & \mathbb{P}\{X_n \in A | X^{n-1} = x^{n-1}, Y^{n-1} = y^{n-1}\} \\ &= \mathbf{P}(\{\mathbf{x} : x_n \in A\} | \mathbf{y} | \mathcal{B}(\mathcal{X}_{0,n-1})) = p_n(A_n; x^{n-1}, y^{n-1}) - a.s. \\ & \mathbb{P}\{Y_n \in B | Y^{n-1} = y^{n-1}, X^n = x^n\} \\ &= \mathbf{Q}(\{\mathbf{y} : y_n \in B\} | \mathbf{x} | \mathcal{B}(\mathcal{Y}_{0,n-1})) = q_n(B_n; y^{n-1}, x^n) - a.s. \end{aligned}$$

Note that $p_n(\cdot; \cdot, \cdot) \in \mathcal{Q}(\mathcal{X}_n; \mathcal{X}_{0,n-1} \times \mathcal{Y}_{0,n-1})$ and $q_n(\cdot; \cdot, \cdot) \in \mathcal{Q}(\mathcal{Y}_n; \mathcal{Y}_{0,n-1} \times \mathcal{X}_{0,n})$ are stochastic kernels [14], determined from $\mathbf{P}(\cdot|\cdot)$ and $\mathbf{Q}(\cdot|\cdot)$, respectively, (e.g., related via (4), (6)). The distribution of RV's $\{(X_i, Y_i) : i \in \mathbb{N}\}$ is defined by

$$\begin{aligned} & P\{X_0 \in A_0, Y_0 \in B_0, \dots, X_n \in A_n, Y_n \in B_n\} \\ &= \int_{A_0} p_0(dx_0) \int_{B_0} q_0(dy_0; x_0) \dots \int_{B_n} q_n(dy_n; y^{n-1}, x^n) \end{aligned}$$

Hence, for any $\mathbf{P}(\cdot|\cdot)$ and $\mathbf{Q}(\cdot|\cdot)$ satisfying consistency conditions there exist a probability space and a sequence of RV's $\{(X_i, Y_i) : i \in \mathbb{N}\}$ defined on it, whose joint probability distribution is defined uniquely via $\mathbf{P}(\cdot|\cdot)$ and $\mathbf{Q}(\cdot|\cdot)$.

III. DIRECTED INFORMATION PROPERTIES AND COMPACTNESS

In this section, directed information $I(X^n \rightarrow Y^n)$ will be defined via relative entropy, using the basic measures $\mathbf{P}(\cdot|\mathbf{y})$ and $\mathbf{Q}(\cdot|\mathbf{x})$, and identify its properties. Define

$$\mathcal{Q}^{\mathbf{C1}}(\mathcal{X}^{\mathbb{N}}; \mathcal{Y}^{\mathbb{N}}) \triangleq \left\{ \mathbf{P}(\cdot|\mathbf{y}) \in \mathcal{M}_1(\mathcal{X}^{\mathbb{N}}) : \mathbf{P}(\cdot|\mathbf{y}) \text{ are regular probability measures and satisfy consistency condition } \mathbf{C1} \right\}.$$

$$\mathcal{Q}^{\mathbf{C2}}(\mathcal{Y}^{\mathbb{N}}; \mathcal{X}^{\mathbb{N}}) \triangleq \left\{ \mathbf{Q}(\cdot|\mathbf{x}) \in \mathcal{M}_1(\mathcal{Y}^{\mathbb{N}}) : \mathbf{Q}(\cdot|\mathbf{x}) \text{ are regular probability measures and satisfy consistency condition } \mathbf{C2} \right\}.$$

Given conditional distributions $\mathbf{P}(\cdot|\cdot) \in \mathcal{Q}^{\mathbf{C1}}(\mathcal{X}^{\mathbb{N}}; \mathcal{Y}^{\mathbb{N}})$ and $\mathbf{Q}(\cdot|\cdot) \in \mathcal{Q}^{\mathbf{C2}}(\mathcal{Y}^{\mathbb{N}}; \mathcal{X}^{\mathbb{N}})$ define the following measures.

P1: The joint distribution on $\mathcal{X}^{\mathbb{N}} \times \mathcal{Y}^{\mathbb{N}}$ defined uniquely by

$$\begin{aligned} & (\bar{\mathbf{P}}_{0,n} \otimes \bar{\mathbf{Q}}_{0,n})(\times_{i=0}^n A_i \times B_i), A_i \in \mathcal{B}(\mathcal{X}_i), B_i \in \mathcal{B}(\mathcal{Y}_i), \\ & \triangleq \mathbb{P}\{X_0 \in A_0, Y_0 \in B_0, \dots, X_n \in A_n, Y_n \in B_n\} \end{aligned} \quad (8)$$

Formally, (8) is written as $(\bar{\mathbf{P}}_{0,n} \otimes \bar{\mathbf{Q}}_{0,n})(dx^n, dy^n)$.

P2: The marginal distributions on $\mathcal{X}^{\mathbb{N}}$ defined uniquely by

$$\begin{aligned} & \mu_{0,n}(\times_{i=0}^n A_i), A_i \in \mathcal{B}(\mathcal{X}_i), 1 \leq i \leq n \\ & \triangleq \mathbb{P}\{X_0 \in A_0, Y_0 \in \mathcal{Y}_0, \dots, X_n \in A_n, Y_n \in \mathcal{Y}_n\}, \\ & = (\bar{\mathbf{P}}_{0,n} \otimes \bar{\mathbf{Q}}_{0,n})(\times_{i=0}^n (A_i \times \mathcal{Y}_i)) \end{aligned}$$

P3: The marginal distributions on $\mathcal{Y}^{\mathbb{N}}$ defined uniquely by

$$\begin{aligned} & \nu_{0,n}(\times_{i=0}^n B_i) \triangleq \mathbb{P}\{X_0 \in \mathcal{X}_0, Y_0 \in B_0, \\ & \dots, X_n \in \mathcal{X}_n, Y_n \in B_n\}, B_i \in \mathcal{B}(\mathcal{Y}_i), 1 \leq i \leq n \\ & = (\bar{\mathbf{P}}_{0,n} \otimes \bar{\mathbf{Q}}_{0,n})(\times_{i=0}^n (\mathcal{X}_i \times B_i)) \end{aligned}$$

P4: The measure $\bar{\Pi}_{0,n} : \mathcal{B}(\mathcal{X}_{0,n}) \odot \mathcal{B}(\mathcal{Y}_{0,n}) \mapsto [0, 1]$ defined uniquely for $A_i \in \mathcal{B}(\mathcal{X}_i), B_i \in \mathcal{B}(\mathcal{Y}_i), 1 \leq i \leq n$ by

$$\bar{\Pi}_{0,n}(\times_{i=0}^n (A_i \times B_i)) \triangleq (\bar{\mathbf{P}}_{0,n} \otimes \nu_{0,n})(\times_{i=0}^n (A_i \times B_i)),$$

P5: The measure $\bar{\Pi}_{0,n} : \mathcal{B}(\mathcal{Y}_{0,n}) \odot \mathcal{B}(\mathcal{X}_{0,n}) \mapsto [0, 1]$ defined uniquely for $A_i \in \mathcal{B}(\mathcal{X}_i), B_i \in \mathcal{B}(\mathcal{Y}_i), 1 \leq i \leq n$ by

$$\bar{\Pi}_{0,n}(\times_{i=0}^n (A_i \times B_i)) \triangleq (\mu_{0,n} \otimes \bar{\mathbf{Q}}_{0,n})(\times_{i=0}^n (A_i \times B_i))$$

A. Directed Information

Let $\mathbf{P}(\cdot|\cdot) \in \mathcal{Q}^{\mathbf{C1}}(\mathcal{X}^{\mathbb{N}}; \mathcal{Y}^{\mathbb{N}})$ and $\mathbf{Q}(\cdot|\cdot) \in \mathcal{Q}^{\mathbf{C2}}(\mathcal{X}^{\mathbb{N}}; \mathcal{Y}^{\mathbb{N}})$. By invoking the definition of directed information, it can be shown (using the chain rule of relative entropy and the relation between absolute continuity of measures) [14] that directed information is equivalently given by

$$I(X^n \rightarrow Y^n) = \mathbb{D}(\bar{\mathbf{P}}_{0,n} \otimes \bar{\mathbf{Q}}_{0,n} \| \bar{\Pi}_{0,n}) \quad (9)$$

$$\begin{aligned} &= \int_{\mathcal{X}_{0,n} \times \mathcal{Y}_{0,n}} \log \left(\frac{\bar{\mathbf{Q}}_{0,n}(dy^n | x^n)}{\nu_{0,n}(dy^n)} \right) (\bar{\mathbf{P}}_{0,n} \otimes \bar{\mathbf{Q}}_{0,n})(dx^n, dy^n) \\ &\equiv \mathbb{I}_{X^n \rightarrow Y^n}(\bar{\mathbf{P}}_{0,n}, \bar{\mathbf{Q}}_{0,n}) \end{aligned} \quad (10)$$

The notation $\mathbb{I}_{X^n \rightarrow Y^n}(\cdot, \cdot)$ indicates the functional dependence of $I(X^n \rightarrow Y^n)$ on $\{\bar{\mathbf{P}}_{0,n}, \bar{\mathbf{Q}}_{0,n}\}$. The investigation of the functional properties of directed information is done via $\mathbb{I}_{X^n \rightarrow Y^n}(\cdot, \cdot)$.

B. Convexity and Concavity of Directed Information

Let $\mathcal{Q}^{\mathbf{C1}}(\mathcal{X}_{0,n}; \mathcal{Y}_{0,n-1})$, $\mathcal{Q}^{\mathbf{C2}}(\mathcal{Y}_{0,n}; \mathcal{X}_{0,n})$ be the restrictions of $\mathcal{Q}^{\mathbf{C1}}(\mathcal{X}^{\mathbb{N}}; \mathcal{Y}^{\mathbb{N}})$ and $\mathcal{Q}^{\mathbf{C2}}(\mathcal{Y}^{\mathbb{N}}; \mathcal{X}^{\mathbb{N}})$, respectively, to cylinder sets with bases over $A_i \in \mathcal{B}(\mathcal{X}_i)$, and $B_i \in \mathcal{B}(\mathcal{Y}_i)$, $i = 0, 1, \dots, n$. These are regular conditional distributions.

Theorem 3.1: Let $\{\mathcal{X}_n, \mathcal{B}(\mathcal{X}_n) : n \in \mathbb{N}\}$, $\{\mathcal{Y}_n, \mathcal{B}(\mathcal{Y}_n) : n \in \mathbb{N}\}$ be Polish spaces. Then

- 1) $\mathcal{Q}^{\mathbf{C1}}(\mathcal{X}_{0,n}; \mathcal{Y}_{0,n-1})$, $\mathcal{Q}^{\mathbf{C2}}(\mathcal{Y}_{0,n}; \mathcal{X}_{0,n})$ are convex sets.

- 2) $\mathbb{I}_{X^n \rightarrow Y^n}(\bar{P}_{0,n}, \bar{Q}_{0,n})$ is a convex functional of $\bar{Q}_{0,n} \in \mathcal{Q}^{C2}(\mathcal{Y}_{0,n}; \mathcal{X}_{0,n})$ for a fixed $\bar{P}_{0,n} \in \mathcal{Q}^{C1}(\mathcal{X}_{0,n}; \mathcal{Y}_{0,n-1})$.
- 3) $\mathbb{I}_{X^n \rightarrow Y^n}(\bar{P}_{0,n}, \bar{Q}_{0,n})$ is a concave functional of $\bar{P}_{0,n} \in \mathcal{Q}^{C1}(\mathcal{X}_{0,n}; \mathcal{Y}_{0,n-1})$ for a fixed $\bar{Q}_{0,n} \in \mathcal{Q}^{C2}(\mathcal{Y}_{0,n}; \mathcal{X}_{0,n})$.

Proof: 1) Follows from the convexity of regular conditional distributions, since $\mathcal{Q}^{C1}(\mathcal{X}_{0,n}; \mathcal{Y}_{0,n-1})$, $\mathcal{Q}^{C2}(\mathcal{Y}_{0,n}; \mathcal{X}_{0,n})$ are subsets satisfying consistency condition C1, C2, respectively. 2), 3), follow from log-sum formulae. ■

C. Lower semicontinuity-Continuity of Directed Information

This part discusses the lower-semicontinuity and continuity of directed information as a functional of $\bar{P}_{0,n}(\cdot|y^{n-1}) \in \mathcal{Q}^{C1}(\mathcal{X}_{0,n}; \mathcal{Y}_{0,n-1})$ and $\bar{Q}_{0,n}(\cdot|x^n) \in \mathcal{Q}^{C2}(\mathcal{Y}_{0,n}; \mathcal{X}_{0,n})$, with respect to the topology of weak convergence of probability measures. Before establishing the main results, sufficient conditions for weak compactness of the set of measures $\mathcal{Q}^{C1}(\mathcal{X}_{0,n}; \mathcal{Y}_{0,n-1})$, $\mathcal{Q}^{C2}(\mathcal{Y}_{0,n}; \mathcal{X}_{0,n})$, and joint and marginal measures are given.

Theorem 3.2:

Part A. Let $\mathcal{Y}_{0,n}$ be a compact Polish space and $\mathcal{X}_{0,n}$ a Polish space. Assume $\bar{P}_{0,n}(\cdot|y^{n-1}) \in \mathcal{Q}^{C1}(\mathcal{X}_{0,n}; \mathcal{Y}_{0,n-1})$ satisfy the following condition.

CA: For all $g(\cdot) \in BC(\mathcal{X}_{0,n})$, where $BC(\mathcal{X}_{0,n})$ denotes the set of bounded continuous real-valued functions on $\mathcal{X}_{0,n}$,

$$(x^{n-1}, y^{n-1}) \mapsto \int_{\mathcal{X}_n} g(x) p_n(dx; x^{n-1}, y^{n-1}) \in \mathbb{R} \quad (11)$$

is jointly continuous in $(x^{n-1}, y^{n-1}) \in \mathcal{X}_{0,n-1} \times \mathcal{Y}_{0,n-1}$. Then the following weak convergence results hold.

- A1) Let $\bar{P}_{0,n}(\cdot|y^{n-1}) \in \mathcal{Q}^{C1}(\mathcal{X}_{0,n}; \mathcal{Y}_{0,n-1})$ and $\{\bar{Q}_{0,n}^\alpha(\cdot|x^n)\}_{\alpha \geq 1} \in \mathcal{Q}^{C2}(\mathcal{Y}_{0,n}; \mathcal{X}_{0,n})$. Then the joint measure $(\bar{P}_{0,n} \otimes \bar{Q}_{0,n}^\alpha)(dx^n, dy^n) \xrightarrow{w} (\bar{P}_{0,n} \otimes \bar{Q}_{0,n}^0)(dx^n, dy^n)$, where $\bar{Q}_{0,n}^0(\cdot|x^n) \in \mathcal{Q}^{C2}(\mathcal{Y}_{0,n}; \mathcal{X}_{0,n})$.
- A2) Let $\bar{P}_{0,n}(\cdot|y^{n-1}) \in \mathcal{Q}^{C1}(\mathcal{X}_{0,n}; \mathcal{Y}_{0,n-1})$ and $\{\bar{Q}_{0,n}^\alpha(\cdot|x^n)\}_{\alpha \geq 1} \in \mathcal{Q}^{C2}(\mathcal{Y}_{0,n}; \mathcal{X}_{0,n})$ and define the family of joint measures $\{(\bar{P}_{0,n} \otimes \bar{Q}_{0,n}^\alpha)(dx^n, dy^n)\}_{\alpha \geq 1}$ having marginals $\{\nu_{0,n}^\alpha\}_{\alpha \geq 1}$ on $\mathcal{Y}_{0,n}$ and $\{\mu_{0,n}^\alpha\}_{\alpha \geq 1}$ on $\mathcal{X}_{0,n}$. Then $\nu_{0,n}^\alpha(dy^n) \xrightarrow{w} \nu_{0,n}^0(dy^n)$ and $\mu_{0,n}^\alpha(dx^n) \xrightarrow{w} \mu_{0,n}^0(dx^n)$ where $\nu_{0,n}^0 \in \mathcal{M}_1(\mathcal{Y}_{0,n})$ and $\mu_{0,n}^0 \in \mathcal{M}_1(\mathcal{X}_{0,n})$ are the marginals of $(\bar{P}_{0,n} \otimes \bar{Q}_{0,n}^0)(dx^n, dy^n)$.
- A3) The sets of measures $\mathcal{Q}^{C1}(\mathcal{X}_{0,n}; \mathcal{Y}_{0,n-1})$, and $\mathcal{Q}^{C2}(\mathcal{Y}_{0,n}; \mathcal{X}_{0,n})$ are weakly compact.
- A4) Let $\bar{P}_{0,n}(\cdot|y^{n-1}) \in \mathcal{Q}^{C1}(\mathcal{X}_{0,n}; \mathcal{Y}_{0,n-1})$, $\{\bar{Q}_{0,n}^\alpha(\cdot|x^n)\}_{\alpha \geq 1} \in \mathcal{Q}^{C2}(\mathcal{Y}_{0,n}; \mathcal{X}_{0,n})$, and $\{\nu_{0,n}^\alpha\}_{\alpha \geq 1}$ are the marginals of $\{(\bar{P}_{0,n} \otimes \bar{Q}_{0,n}^\alpha)(dx^n, dy^n)\}_{\alpha \geq 1}$. Then $\bar{\Pi}^\alpha(dx^n, dy^n) \equiv \bar{P}_{0,n}(dx^n|dy^{n-1}) \otimes \nu_{0,n}^\alpha(dy^n) \xrightarrow{w} \bar{P}_{0,n}(dx^n|dy^{n-1}) \otimes \nu_{0,n}^0(dy^n) \equiv$

$\bar{\Pi}^0(dx^n, dy^n)$, where $\nu_{0,n}^0 \in \mathcal{M}_1(\mathcal{Y}_{0,n})$ is the limit of $\nu_{0,n}^\alpha \in \mathcal{M}_1(\mathcal{Y}_{0,n})$.

Part B. Let $\mathcal{X}_{0,n}$ be a compact Polish space and $\mathcal{Y}_{0,n}$ a Polish space. Assume $\bar{Q}_{0,n}(\cdot|x^n) \in \mathcal{Q}^{C2}(\mathcal{Y}_{0,n}; \mathcal{X}_{0,n})$ satisfy the following condition.

CB: For all $h(\cdot) \in BC(\mathcal{Y}_{0,n})$, the function

$$(x^n, y^{n-1}) \mapsto \int_{\mathcal{Y}_n} h(y) q_n(dy; y^{n-1}, x^n) \in \mathbb{R} \quad (12)$$

is jointly continuous in $(x^n, y^{n-1}) \in \mathcal{X}_{0,n} \times \mathcal{Y}_{0,n-1}$.

The statements of **Part A** hold by interchanging $\bar{Q}_{0,n}(\cdot|x^n)$ by $\bar{P}_{0,n}(\cdot|y^{n-1})$, $\nu_{0,n}(dy^n)$ by $\mu_{0,n}(dx^n)$, $\bar{\Pi}(dx^n, dy^n)$ by $\bar{\Pi}(dx^n, dy^n)$.

Proof: The proof is quite lengthy and it is based on Prohorov's theorem relating tightness and weak compactness of a family of probability measures [14]. ■

The results of Theorem 3.2 are sufficient to establish lower semicontinuity of directed information $I(X^n \rightarrow Y^n) \equiv \mathbb{I}_{X^n \rightarrow Y^n}(\bar{P}_{0,n}, \bar{Q}_{0,n})$.

Theorem 3.3: 1) Suppose the conditions in Theorem 3.2, **Part A** hold. Then $\mathbb{I}_{X^n \rightarrow Y^n}(\bar{P}_{0,n}, \bar{Q}_{0,n})$ is lower semicontinuous on $\bar{Q}_{0,n} \in \mathcal{Q}^{C2}(\mathcal{Y}_{0,n}; \mathcal{X}_{0,n})$ for fixed $\bar{P}_{0,n} \in \mathcal{Q}^{C1}(\mathcal{X}_{0,n}; \mathcal{Y}_{0,n-1})$.

2) Suppose the conditions in Theorem 3.2, **Part B** hold. Then $\mathbb{I}_{X^n \rightarrow Y^n}(\bar{P}_{0,n}, \bar{Q}_{0,n})$ is lower semicontinuous on $\bar{P}_{0,n} \in \mathcal{Q}^{C1}(\mathcal{X}_{0,n}; \mathcal{Y}_{0,n-1})$ for fixed $\bar{Q}_{0,n} \in \mathcal{Q}^{C2}(\mathcal{Y}_{0,n}; \mathcal{X}_{0,n})$.

Proof: Utilizes (9), Theorem 3.2, and lower semicontinuity of relative entropy. ■

For capacity problems, it is desirable to identify conditions so that $\mathbb{I}_{X^n \rightarrow Y^n}(\bar{P}_{0,n}, \bar{Q}_{0,n})$ as a function of $\bar{P}_{0,n}$ for fixed $\bar{Q}_{0,n}$ is either upper semicontinuous or continuous. Continuity of directed information is established by generalizing the derivation in [15].

Theorem 3.4: Suppose the conditions in Theorem 3.2, **Part B** hold. Consider a forward channel $\bar{Q}_{0,n}(\cdot|x^n) \in \mathcal{Q}^{C2}(\mathcal{Y}_{0,n}; \mathcal{X}_{0,n})$, and a closed family of feedback channels $\bar{Q}^{C1}(\mathcal{X}_{0,n}; \mathcal{Y}_{0,n-1}) \subseteq \mathcal{Q}^{C1}(\mathcal{X}_{0,n}; \mathcal{Y}_{0,n-1})$. Suppose there exists a family of measures $\bar{\nu}_{0,n}(dy^n)$ on $(\mathcal{Y}_{0,n}, \mathcal{B}(\mathcal{Y}_{0,n}))$ such that $\bar{Q}_{0,n}(\cdot|x^n) \ll \bar{\nu}_{0,n}(dy^n)$ with Radon-Nikodym derivatives $\xi_{\bar{\nu}_{0,n}}(x^n, y^n) \triangleq \frac{\bar{Q}_{0,n}(\cdot|x^n)}{\bar{\nu}_{0,n}(\cdot)}(y^n)$. Furthermore, suppose the following conditions hold.

A. The family of Radon-Nikodym derivatives $\xi_{\bar{\nu}_{0,n}}(x^n, y^n)$ is continuous on $\mathcal{X}_{0,n} \times \mathcal{Y}_{0,n}$, and $\xi_{\bar{\nu}_{0,n}}(x^n, y^n) \log \xi_{\bar{\nu}_{0,n}}(x^n, y^n)$ is uniformly integrable over $\{\bar{\nu}_{0,n} \otimes \bar{P}_{0,n} : \bar{P}_{0,n} \in \bar{Q}^{C1}(\mathcal{X}_{0,n}; \mathcal{Y}_{0,n-1})\}$.

B. For a fixed $y^n \in \mathcal{Y}_{0,n}$, the Radon-Nikodym derivative $\xi_{\bar{\nu}_{0,n}}(x^n, y^n)$ is uniformly integrable over $\bar{Q}^{C1}(\mathcal{X}_{0,n}; \mathcal{Y}_{0,n-1})$. Then, the directed information $\mathbb{I}_{X^n \rightarrow Y^n}(\bar{P}_{0,n}, \bar{Q}_{0,n})$ as a functional of $\{\bar{P}_{0,n}, \bar{Q}_{0,n}\} \in \bar{Q}^{C1}(\mathcal{X}_{0,n}; \mathcal{Y}_{0,n-1}) \times \bar{Q}^{C2}(\mathcal{Y}_{0,n}; \mathcal{X}_{0,n})$ is bounded and weakly continuous over $\bar{Q}^{C1}(\mathcal{X}_{0,n}; \mathcal{Y}_{0,n-1})$.

Proof: The derivation invokes (9) and generalizes related results in [15]. ■

IV. EXTREMUM PROBLEMS OF DIRECTED INFORMATION

In this section, sufficient conditions are given for the existence of the extremum problems $C^f(P)$ and $R^c(D)$ (mentioned in introduction).

A. Existence of Capacity Achieving Distribution

Consider a communication channel with memory and feedback $\vec{Q}_{0,n}(\cdot|x^n) \in \mathcal{Q}^{C^2}(\mathcal{Y}_{0,n}; \mathcal{X}_{0,n})$ with power constraints

$$\begin{aligned} \overleftarrow{\mathcal{P}}(P) \triangleq & \{ \overleftarrow{P}_{0,n}(\cdot|y^{n-1}) \in \mathcal{Q}^{C^1}(\mathcal{X}_{0,n}; \mathcal{Y}_{0,n-1}) : \\ & \int_{\mathcal{X}_{0,n} \times \mathcal{Y}_{0,n}} g_{0,n}(x^n, y^{n-1})(\overleftarrow{Q}_{0,n} \otimes \overrightarrow{P}_{0,n})(dx^n, dy^n) \leq P \} \end{aligned}$$

where for any $n \in \mathbb{N}$, $g_{0,n} : \mathcal{X}_{0,n} \times \mathcal{Y}_{0,n-1} \mapsto [0, \infty]$ is Borel measurable. In the absence of any power constraints the set of input conditional distributions is $\mathcal{Q}^{C^1}(\mathcal{X}_{0,n}; \mathcal{Y}_{0,n-1})$.

The finite horizon maximization of directed information over $\overleftarrow{\mathcal{P}}(P)$ or $\mathcal{Q}^{C^1}(\mathcal{X}_{0,n}; \mathcal{Y}_{0,n-1})$ (e.g., with or without power constraints) is defined by

$$C_{0,n}^f \triangleq \sup_{\substack{\overleftarrow{P}_{0,n}(\cdot|y^{n-1}) \in \overleftarrow{\mathcal{P}}(P) \\ \text{or } \mathcal{Q}^{C^1}(\mathcal{X}_{0,n}; \mathcal{Y}_{0,n-1})}} \mathbb{I}_{X^n \rightarrow Y^n}(\overleftarrow{P}_{0,n}, \overrightarrow{Q}_{0,n}) \quad (13)$$

The next theorem establishes existence of the maximizer.

Theorem 4.1: Suppose that the assumptions of Theorem 3.2, **Part B** are satisfied.

1. The set $\mathcal{Q}^{C^1}(\mathcal{X}_{0,n}; \mathcal{Y}_{0,n-1})$ is compact.
2. The set $\overleftarrow{\mathcal{P}}(P)$ is a closed subset of $\mathcal{Q}^{C^1}(\mathcal{X}_{0,n}; \mathcal{Y}_{0,n-1})$.
3. If in addition the assumptions of Theorem 3.4 are satisfied (here the assumption on $\mathcal{Q}^{C^1}(\mathcal{X}_{0,n}; \mathcal{Y}_{0,n-1})$ is satisfied by 1. and 2.) then $C_{0,n}^f$ has a maximum in $\mathcal{Q}^{C^1}(\mathcal{X}_{0,n}; \mathcal{Y}_{0,n-1})$ (without constraints) or in $\overleftarrow{\mathcal{P}}(P)$ (with power constraints).

Proof: 1. Utilize the fact that probability measures on compact Polish spaces are compact. 2. Utilize the fact that closed subset of weakly compact set is compact. 3. Follows from Weierstrass theorem. ■

B. Existence of Causal Rate Distortion Achieving Distribution

Consider a reconstruction channel $\vec{Q}_{0,n}(\cdot|x^n) \in \mathcal{Q}^{C^2}(\mathcal{Y}_{0,n}; \mathcal{X}_{0,n})$ and a fixed source distribution $\mu_{0,n}(dx^n) \in \mathcal{M}_1(\mathcal{X}_{0,n})$ define the fidelity constraint by

$$\begin{aligned} \vec{\mathcal{Q}}(D) \triangleq & \{ \vec{Q}_{0,n}(\cdot|x^n) \in \mathcal{Q}^{C^2}(\mathcal{Y}_{0,n}; \mathcal{X}_{0,n}) : \\ & \int_{\mathcal{X}_{0,n} \times \mathcal{Y}_{0,n}} d_{0,n}(x^n, y^n)(\mu_{0,n} \otimes \vec{Q}_{0,n})(dx^n, dy^n) \leq D \} \end{aligned} \quad (14)$$

where $D \geq 0$, and for each $n \in \mathbb{N}$ the distortion function $d_{0,n} : \mathcal{X}_{0,n} \times \mathcal{Y}_{0,n} \mapsto [0, \infty)$ is Borel measurable.

The finite horizon minimization of directed information over $\vec{\mathcal{Q}}(D)$ is defined by

$$R_{0,n}^c(D) \triangleq \inf_{\vec{Q}_{0,n}(\cdot|x^n) \in \vec{\mathcal{Q}}(D)} \mathbb{I}_{X^n \rightarrow Y^n}(\mu_{0,n}, \vec{Q}_{0,n}) \quad (15)$$

The next theorem establishes existence of the minimizer.

Theorem 4.2: Suppose that the assumptions of Theorem 3.2, **Part A** are satisfied.

1. The set $\mathcal{Q}^{C^2}(\mathcal{Y}_{0,n}; \mathcal{X}_{0,n})$ is compact.
2. The set $\vec{\mathcal{Q}}(D)$ is a closed subset of $\mathcal{Q}^{C^2}(\mathcal{Y}_{0,n}; \mathcal{X}_{0,n})$.
3. $R_{0,n}^c(D)$ has a minimum in $\vec{\mathcal{Q}}(D)$.

Proof: The proof is based on generalizing the derivation in [13] to n -fold convolution measures. It is done by induction. ■

V. CONCLUSION

In this paper we have provided a general framework through which the properties of mutual information are extended to directed information on Polish spaces. The existence of solutions to capacity problems with memory and feedback, and to lossy causal data compression problems is shown.

REFERENCES

- [1] H. Marko, "The bidirectional communication theory—a generalization of information theory," *IEEE Transactions on Communications*, vol. 21, no. 12, pp. 1345 – 1351, dec 1973.
- [2] J. L. Massey, "Causality, feedback and directed information," in *International Symposium on Information Theory and its Applications (ISITA '90)*, Nov. 27–30 1990, pp. 303 – 305.
- [3] S. Tatikonda and S. Mitter, "The capacity of channels with feedback," *IEEE Transactions on Information Theory*, vol. 55, no. 1, pp. 323 – 349, jan 2009.
- [4] S. C. Tatikonda, "Control over communication constraints," Ph.D. dissertation, Mass. Inst. of Tech. (M.I.T.), Cambridge, MA, 2000.
- [5] J. Chen and T. Berger, "The capacity of finite-state markov channels with feedback," *IEEE Transactions on Information Theory*, vol. 51, no. 3, pp. 780 – 798, march 2005.
- [6] R. Venkataramanan, "Information-theoretic results on communication problems with feed-forward and feedback," Ph.D. dissertation, University of Michigan-Ann Arbor, December 2008.
- [7] C. D. Charalambous, P. A. Stavrou, and C. K. Kourtellis, "Causal rate distortion function on abstract alphabets and optimal reconstruction kernel," *CoRR*, vol. abs/1102.3294, 2011. [Online:] Available at <http://arxiv.org/abs/1102.3294v1>.
- [8] G. Kramer, "Directed information for channels with feedback," Ph.D. dissertation, Swiss Federal Institute of Technology (ETH), December 1998.
- [9] —, "Capacity results for the discrete memoryless network," *IEEE Transactions on Information Theory*, vol. 49, no. 1, pp. 4 – 21, jan 2003.
- [10] T. S. Han and S. Verdú, "Approximation theory of output statistics," *IEEE Transactions on Information Theory*, vol. 39, no. 3, pp. 752 – 772, may 1993.
- [11] T. M. Cover and J. A. Thomas, *Elements of Information Theory*, 2nd ed. John Wiley & Sons, Inc., Hoboken, New Jersey, 2006.
- [12] I. Csiszár, "Arbitrarily varying channels with general alphabets and states," *IEEE Transactions on Information Theory*, vol. 38, no. 6, pp. 1725 – 1742, nov 1992.
- [13] —, "On an extremum problem of information theory," *Studia Scientiarum Mathematicarum Hungarica*, vol. 9, pp. 57 – 71, 1974.
- [14] P. Dupuis and R. S. Ellis, *A weak Convergence Approach to the Theory of Large Deviations*. John Wiley & Sons, Inc., New York, 1997.
- [15] M. Fozunbal, S. McLaughlin, and R. Schaefer, "Capacity analysis for continuous-alphabet channels with side information, part I: A general framework," *IEEE Transactions on Information Theory*, vol. 51, no. 9, pp. 3075 – 3085, sept 2005.